

# Continuous Dependence for Backward Parabolic Operators with Log-Lipschitz Coefficients

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## Abstract

We prove continuous dependence on Cauchy data for a backward parabolic operator whose coefficients are Log-Lipschitz continuous in time.

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## 1 Introduction

In this paper we prove a new continuous dependence result for solutions of the Cauchy problem associated to the backward parabolic operator

$$P = \partial_t + \sum_{i,j} \partial_{x_i} (a_{i,j}(t, x) \partial_{x_j}) + \sum_j b_j(t, x) \partial_{x_j} + c(t, x) \quad (1.1)$$

on the strip  $[0, T] \times \mathbb{R}^n$ .

It is well known that the Cauchy problem for (1.1), when the data are given on  $\{t = 0\}$  and the matrix  $(a_{i,j})_{i,j=1}^n$  is supposed to be symmetric and positive definite, is an ill-posed problem: due to the smoothing effect of forward parabolic operators, the existence of the solutions is not ensured for all choice of data. Concerning uniqueness, we can say that an important role is played by the functional space in which the uniqueness property is looked for. In fact a classical result of Tychonoff in [13] proves that there exists a function  $u \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$  satisfying  $\partial_t u - \Delta u \equiv 0$  in  $\mathbb{R} \times \mathbb{R}^n$ ,  $u(0, \cdot) \equiv 0$  in  $\mathbb{R}^n$ , but  $u \not\equiv 0$  in all open subset of  $\mathbb{R} \times \mathbb{R}^n$ . On the other hand, in [9], Lions and Malgrange proved that  $P$  enjoys the uniqueness property in  $\mathcal{H}_1 := H^1([0, T], L^2(\mathbb{R}^n)) \cap L^2([0, T], H^2(\mathbb{R}^n))$ , provided the coefficients  $a_{i,j}$ 's are sufficiently smooth with respect to  $x$  and Lipschitz continuous with respect to  $t$ .

If one considers the Cauchy problem for (1.1) as an inverse problem (namely: a final time problem) for a forward parabolic operator (see [7, Ch. 3]), it turns out that uniqueness is a very weak property. Indeed it furnishes only a qualitative feature of the solutions and gives no useful information for computational purposes.

In his celebrated paper [8], John introduced the notion of *well-behaved problem*, which is now typical in the context of ill-posed problems. According to John a problem is *well-behaved* if “only a fixed percentage of the significant digits need be lost in determining the solution from the data” [8, p. 552]. More precisely we may say that a problem is well-behaved if its solutions in a space  $\mathcal{H}$  depend Hölder continuously on the data belonging to a space  $\mathcal{K}$ , provided they satisfy a prescribed bound.

In their paper [1], Agmon and Nirenberg proved, among other things, that the Cauchy problem for (1.1) is well-behaved in  $\mathcal{E} := C^0([0, T], L^2(\mathbb{R}^n)) \cap C^0([0, T[, H^1(\mathbb{R}^n)) \cap C^1([0, T[, L^2(\mathbb{R}^n))$  with data in  $L^2(\mathbb{R}^n)$ , provided the coefficients  $a_{i,j}$ ’s are sufficiently smooth with respect to  $x$  and Lipschitz continuous with respect to  $t$ . In order to achieve their result, which is stated in a very general and abstract setting, they developed the so called *logarithmic convexity technique*. The main step consists in proving that the function  $t \mapsto \log \|u(t, \cdot)\|_{L^2}$  is convex for every solution  $u \in \mathcal{E}$  of (1.1). In the same year Glagoleva [5] obtained essentially the same result for a concrete operator like (1.1) with time independent coefficients. Her proof rests on energy estimates obtained through integration by parts. Some years later Hurd [6] developed the technique of Glagoleva so as to cover the case of a general operator of type (1.1), with coefficients depending Lipschitz continuously on time. The results of [1, 5, 6] can be summarized as follows:

*For every  $T' \in ]0, T[$  and  $D > 0$  there exist  $\rho > 0$ ,  $0 < \delta < 1$  and  $M > 0$  such that, if  $u \in \mathcal{E}$  is a solution of  $Pu \equiv 0$  on  $[0, T]$  with  $\|u(0, \cdot)\|_{L^2} \leq \rho$  and  $\|u(t, \cdot)\|_{L^2} \leq D$  on  $[0, T]$ , then*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M \|u(0, \cdot)\|_{L^2}^\delta.$$

*The constants  $\rho$ ,  $M$  and  $\delta$  depend only on  $T'$  and  $D$ , on the ellipticity constant of  $P$ , on the  $L^\infty$  norms of the coefficients  $a_{i,j}$ ’s,  $b_i$ ’s,  $c$  and of their spatial derivatives, and on the Lipschitz constant of the coefficients  $a_{i,j}$ ’s with respect to time.*

In [9, 1, 6], Lipschitz continuity of the coefficients  $a_{i,j}$ ’s with respect to time plays an essential role. The possibility of replacing Lipschitz continuity by simple continuity was ruled out by Miller [11] and more recently by Mandache [10]. They constructed examples of operators of the form (1.1) which do not enjoy the uniqueness property in  $\mathcal{H}_1$ . In the example of Miller the coefficients  $a_{i,j}$ ’s are Hölder continuous in time, while in the more refined example of Mandache the modulus of continuity  $\bar{\mu}$  of the coefficients  $a_{i,j}$ ’s with respect to time is such that  $\int_0^1 (1/\bar{\mu}(s)) ds < +\infty$ . On the other hand, in [4] the authors of the present paper

proved that, if  $\bar{\mu}$  satisfies the *Osgood condition*  $\int_0^1 (1/\bar{\mu}(s))ds = +\infty$ , then the operator  $P$  enjoys the uniqueness property in  $\mathcal{H}_1$ . Therefore it would be natural to conjecture that if the Osgood condition is satisfied, then the Cauchy problem for (1.1) is well-behaved in  $\mathcal{E}$  with data in  $L^2(\mathbb{R}^n)$ . Unfortunately this is not true. Let  $\mu(s) := s(1 + |\log s|)$ . A function whose modulus of continuity is  $\mu$  is called *Log-Lipschitz continuous*. Obviously  $\mu$  satisfies the Osgood condition. In the Appendix, we show that it is possible to construct:

- a sequence  $(L_n)_{n \in \mathbb{N}}$  of backward uniformly parabolic operators with space-periodic uniformly Log-Lipschitz continuous coefficients in the principal part and space-periodic uniformly bounded coefficients in lower order terms;
- a sequence  $(u_n)_{n \in \mathbb{N}}$  of space-periodic smooth uniformly bounded solutions of  $L_n u_n = 0$  on  $[0, 1] \times \mathbb{R}^2$ ;
- a sequence  $(t_n)_{n \in \mathbb{N}}$  of real numbers, with  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ ;

such that

$$\lim_{n \rightarrow \infty} \|u_n(0, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{\|u_n(t_n, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}}{\|u_n(0, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}^\delta} = +\infty,$$

for every  $\delta > 0$ .

Therefore it is not possible to obtain a result similar to that of Hurd or Agmon and Nirenberg if Lipschitz continuity is replaced by Log-Lipschitz continuity.

If the coefficients  $a_{i,j}$ 's are Log-Lipschitz continuous in time, we are able to prove a weaker continuous dependence result. Our main result can be stated as follows:

*For every  $T' \in ]0, T[$  and  $D > 0$  there exist  $\rho > 0$ ,  $0 < \delta < 1$  and  $M, N > 0$  such that, if  $u \in \mathcal{E}$  is a solution of  $Pu \equiv 0$  on  $[0, T]$  with  $\|u(0, \cdot)\|_{L^2} \leq \rho$  and  $\|u(t, \cdot)\|_{L^2} \leq D$  on  $[0, T]$ , then*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M e^{-N |\log \|u(0, \cdot)\|_{L^2}|^\delta}.$$

*The constants  $\rho$ ,  $M$ ,  $N$  and  $\delta$  depend only on  $T'$  and  $D$ , on the ellipticity constant of  $P$ , on the  $L^\infty$  norms of the coefficients  $a_{i,j}$ 's,  $b_i$ 's,  $c$  and of their spatial derivatives, and on the Log-Lipschitz constant of the coefficients  $a_{ij}$ 's with respect to time.*

As a consequence, going back to John's terminology, if one denotes by  $\phi(n)$  the number of digits of the  $L^2$  norm of the data which are necessary to determine  $n$  digits of the  $L^2$  norm of the solution, one has that  $\phi(n)$  grows at most polynomially in  $n$ .

Our proof relies on weighted energy estimates similar to those of Glagoleva and Hurd. In order to overcome the obstructions created by the lack of time differentiability of the coefficients  $a_{i,j}$ 's, we exploit a microlocal approximation procedure originally developed by Colombini and Lerner in [3] in the study of the Cauchy problem for hyperbolic operators having Log-Lipschitz coefficients.

The plan of the paper is the following. In Section 2 we introduce notations and we state our results: Theorem 1 contains the weighted energy estimates that we mentioned above; Theorem 2 is a local continuous dependence result; Theorem 3 is a global continuous dependence result. Section 3 is devoted to the proof of Theorem 1, while Section 4 is devoted to the proofs of Theorems 2 and 3. Finally, in the Appendix we outline the construction of a counterexample to Hölder continuous dependence.

## 2 Results

### 2.1 Notations

We consider the following backward parabolic equation

$$\partial_t u + \sum_{i,j} \partial_{x_i} (a_{i,j}(t,x) \partial_{x_j} u) + \sum_j b_j(t,x) \partial_{x_j} u + c(t,x) u = 0 \quad (2.1)$$

on the strip  $[0, T] \times \mathbb{R}^n \ni (t, x)$ . We suppose that

- for all  $(t, x) \in [0, T] \times \mathbb{R}^n$  and for all  $i, j = 1 \dots n$ ,

$$a_{i,j}(t, x) = a_{j,i}(t, x);$$

- there exists  $k > 0$  such that, for all  $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ ,

$$k|\xi|^2 \leq \sum_{i,j} a_{i,j}(t, x) \xi_i \xi_j \leq k^{-1}|\xi|^2;$$

- for all  $i, j = 1, \dots, n$ ,  $a_{i,j} \in \text{LogLip}([0, T], L^\infty(\mathbb{R}^n)) \cap L^\infty([0, T], C_b^2(\mathbb{R}^n))$  and  $b_j, c \in L^\infty([0, T], C_b^2(\mathbb{R}^n))$ .

We set

$$A_{LL} := \sup \left\{ \frac{|a_{i,j}(t, x) - a_{i,j}(s, x)|}{|t - s|(1 + |\log |t - s||)} : i, j = 1, \dots, n, x \in \mathbb{R}^n, \right. \\ \left. t, s \in [0, T], 0 < |t - s| \leq 1 \right\},$$

$$A := \sup \{ |\partial_x^\alpha a_{i,j}(t, x)| : i, j = 1, \dots, n, \alpha \in \mathbb{N}^n, |\alpha| \leq 2, \\ (t, x) \in [0, T] \times \mathbb{R}^n \},$$

$$B := \sup \{ |\partial_x^\alpha b_j(t, x)| : j = 1, \dots, n, \alpha \in \mathbb{N}^n, |\alpha| \leq 2, \\ (t, x) \in [0, T] \times \mathbb{R}^n \},$$

$$C := \sup \{ |\partial_x^\alpha c(t, x)| : \alpha \in \mathbb{N}^n, |\alpha| \leq 2, (t, x) \in [0, T] \times \mathbb{R}^n \}.$$

## 2.2 Weight function

For  $s > 0$ , let  $\mu(s) = s(1 + |\log s|)$ . For  $\tau \geq 1$  we define

$$\theta(\tau) := \int_{1/\tau}^1 \frac{1}{\mu(s)} ds = \log(1 + \log \tau).$$

The function  $\theta : [1, +\infty[ \rightarrow [0, +\infty[$  is bijective and strictly increasing. For  $y \in ]0, 1]$  and  $\lambda > 1$ , we set  $\psi_\lambda(y) := \theta^{-1}(-\lambda \log y) = \exp(y^{-\lambda} - 1)$  and we define

$$\Phi_\lambda(y) := \int_1^y \psi_\lambda(z) dz.$$

The function  $\Phi_\lambda : ]0, 1] \rightarrow ]-\infty, 0]$  is bijective and strictly increasing; moreover it is easy to verify that it satisfies

$$y \Phi_\lambda''(y) = -\lambda (\Phi_\lambda'(y))^2 \mu\left(\frac{1}{\Phi_\lambda'(y)}\right) = -\lambda \Phi_\lambda'(y) \left(1 + \left|\log \frac{1}{\Phi_\lambda'(y)}\right|\right). \quad (2.2)$$

We collect in the following lemma, the proof of which is left to the reader, some interesting and elementary properties of the functions  $\psi_\lambda$  and  $\Phi_\lambda$ .

**Lemma 1.** *Let  $\zeta > 1$ . Then, for  $y \leq 1/\zeta$ ,*

$$\psi_\lambda(\zeta y) = \exp(\zeta^{-\lambda} - 1)(\psi_\lambda(y))^{\zeta^{-\lambda}}. \quad (2.3)$$

*Define  $\Lambda_\lambda(y) := y \Phi_\lambda(1/y)$ . Then the function  $\Lambda_\lambda : [1, +\infty[ \rightarrow ]-\infty, 0]$  is bijective and*

$$\lim_{z \rightarrow -\infty} -\frac{1}{z} \psi_\lambda\left(\frac{1}{\Lambda_\lambda^{-1}(z)}\right) = +\infty. \quad (2.4)$$

## 2.3 Main results

Let  $\mathcal{E} := C^0([0, T], L^2(\mathbb{R}^n)) \cap C^0([0, T[, H^1(\mathbb{R}^n)) \cap C^1([0, T[, L^2(\mathbb{R}^n))$ .

**Theorem 1.** *There exist  $\bar{\lambda} > 1$ ,  $\alpha_1$ ,  $\bar{\gamma}$ ,  $M > 0$  depending only on  $A_{LL}$ ,  $A$ ,  $B$ ,  $C$ ,  $k$ ,  $T$  such that, setting  $\alpha := \max\{\alpha_1, 1/T\}$ ,  $\sigma := 1/\alpha$  and choosing  $\tau \in ]0, \sigma/2[$ , if  $\beta \geq \sigma + \tau$ ,  $\lambda \geq \bar{\lambda}$ ,  $\gamma \geq \bar{\gamma}$  and if  $u \in \mathcal{E}$  is a solution of the equation (2.1), then*

$$\begin{aligned} & \int_0^s e^{2\gamma t} e^{-2\beta \Phi_\lambda((t+\tau)/\beta)} \|u(t, \cdot)\|_{H^{1-\alpha t}}^2 dt \\ & \leq M \left( (s + \tau) e^{2\gamma s} e^{-2\beta \Phi_\lambda((s+\tau)/\beta)} \|u(s, \cdot)\|_{H^{1-\alpha s}}^2 \right. \\ & \quad \left. + \tau \Phi_\lambda'(\tau/\beta) e^{-2\beta \Phi_\lambda(\tau/\beta)} \|u(0, \cdot)\|_{L^2}^2 \right), \end{aligned} \quad (2.5)$$

for all  $0 \leq s \leq \sigma$ .

**Theorem 2.** *There exist  $\sigma > 0$  such that for all  $\bar{\sigma} \in ]0, \sigma/4[$  there exist  $\rho, \bar{M}, N, \delta > 0$  such that, if  $u \in \mathcal{E}$  is a solution of the equation (2.1) with  $\|u(0, \cdot)\|_{L^2} \leq \rho$ , then*

$$\sup_{t \in [0, \bar{\sigma}]} \|u(t, \cdot)\|_{L^2} \leq \bar{M}(1 + \|u(\sigma, \cdot)\|_{L^2})e^{-N(|\log \|u(0, \cdot)\|_{L^2}|)^\delta}. \quad (2.6)$$

**Theorem 3.** *For all  $T' \in ]0, T[$  and for all  $D > 0$  there exist  $\rho', M', N', \delta' > 0$ , depending only on  $A_{LL}, A, B, C, k, T, T', D$ , such that if  $u \in \mathcal{E}$  is a solution of the equation (2.1) with  $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2} \leq D$  and  $\|u(0, \cdot)\|_{L^2} \leq \rho'$ , then*

$$\sup_{t \in [0, T']} \|u(t, \cdot)\|_{L^2} \leq M'e^{-N'|\log \|u(0, \cdot)\|_{L^2}|^{\delta'}}. \quad (2.7)$$

### 3 Proof of Theorem 1

#### 3.1 Dyadic decomposition

We collect here some well known facts on the Littlewood-Paley dyadic decomposition, referring to [2] and [3] for the details. Let  $\varphi \in C^\infty(\mathbb{R})$ ,  $\varphi(x) = 1$  if  $x \leq 1$ ,  $\varphi(x) = 0$  if  $x \geq 2$ ,  $\varphi$  decreasing. We set  $\varphi_0(\xi) = \varphi(|\xi|)$  and, if  $\nu$  is an integer greater than or equal to 1,  $\varphi_\nu(\xi) = \varphi_0(\xi/2^\nu) - \varphi_0(\xi/2^{\nu-1})$ . Let  $w$  be a tempered distribution in  $H^{-\infty}(\mathbb{R}^n)$ ; we define

$$\begin{aligned} w_\nu(x) &= \varphi_\nu(D_x)w(x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} \varphi_\nu(\xi) \hat{w}(\xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int \hat{\varphi}_\nu(y) w(x - y) dy. \end{aligned}$$

For all  $\nu$ ,  $w_\nu$  is an entire analytic function belonging to  $L^2$ . We have

- for all  $\nu \geq 1$

$$2^{\nu-1} \|w_\nu\|_{L^2} \leq \|\nabla_x w_\nu\|_{(L^2)^n} \leq 2^{\nu+1} \|w_\nu\|_{L^2}, \quad (3.1)$$

where the inequality on the right hand side holds also for  $\nu = 0$ ;

- there exist  $K$  such that, for all  $s \in [0, 1]$ ,

$$K \sum_{\nu=0}^{+\infty} 2^{2s\nu} \|w_\nu\|_{L^2}^2 \leq \|w\|_{H^s}^2 \leq \frac{1}{K} \sum_{\nu=0}^{+\infty} 2^{2s\nu} \|w_\nu\|_{L^2}^2; \quad (3.2)$$

- if the function  $u : [0, T[ \rightarrow L^2(\mathbb{R}^n)$  is of class  $C^1$ , then the function  $u_\nu : [0, T[ \rightarrow C_b^m(\mathbb{R}_x^n) \cap H^s(\mathbb{R}_x^n)$  is of class  $C^1$  for all  $s \geq 0$  and for all  $m \in \mathbb{N}$  and, for all  $s \geq 0$  and for all  $\alpha \in \mathbb{N}^n$ ,

$$\partial_t \partial_x^\alpha u_\nu = \partial_x^\alpha \partial_t u_\nu \in C^0([0, T[ \times \mathbb{R}^n) \cap C^0([0, T[, L^2(\mathbb{R}^n)));$$

- if  $a \in C_b^2(\mathbb{R}^n)$ , then there exists  $Q > 0$  such that, for all  $\nu, \mu \in \mathbb{N}$ ,

$$\|[\varphi_\nu(D_x), a]\varphi_\mu(D_x)\|_{\mathcal{L}(L^2, L^2)} \leq \begin{cases} Q 2^{-2\nu} & \text{if } |\mu - \nu| \leq 2, \\ Q 2^{-2\max\{\nu, \mu\}} & \text{if } |\mu - \nu| \geq 3, \end{cases}$$

where  $[\varphi_\nu(D_x), a]w(x) = (\varphi_\nu(D_x)(aw))(x) - a(x)(\varphi_\nu(D_x)u)(x)$  is the commutator between  $\varphi_\nu(D_x)$  and  $a$ ,  $\|\cdot\|_{\mathcal{L}(L^2, L^2)}$  denotes the norm operator in  $L^2$  to  $L^2$  and the constant  $Q$  depends only on  $\|a\|_{C_b^2}$ .

### 3.2 Preliminaries

Let  $u(t, x) \in \mathcal{E}$  be a solution of the equation (2.1). We set

$$w(t, x) := e^{\gamma t} e^{-\beta \Phi_\lambda(\frac{t+\tau}{\beta})} u(t, x),$$

$u_\nu(t, x) := \varphi_\nu(D_x)u(t, x)$ ,  $w_\nu(t, x) := \varphi_\nu(D_x)w(t, x)$ ,  $v_\nu(t, x) := 2^{-\alpha\nu t} w_\nu(t, x)$ , where the constants  $\alpha$ ,  $\lambda$  and  $\gamma$  will be determined later,  $\sigma := 1/\alpha$ ,  $\tau$  is chosen in  $]0, \sigma/2[$  and  $\beta \geq \sigma + \tau$ . The function  $v_\nu$  satisfies

$$\begin{aligned} \partial_t v_\nu = \gamma v_\nu - \sum_{i,j} \partial_{x_i} (a_{i,j}(t, x) \partial_{x_j} v_\nu) - \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu - \alpha(\log 2) \nu v_\nu \\ - \sum_j b_j(t, x) \partial_{x_j} v_\nu - c(t, x) v_\nu + X_\nu(t, x), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} X_\nu(t, x) := - \sum_{i,j} \partial_{x_i} ([\varphi_\nu(D_x), a_{i,j}(t, x)] 2^{-\alpha\nu t} \partial_{x_j} w) \\ - \sum_j [\varphi_\nu(D_x), b_j(t, x)] 2^{-\alpha\nu t} \partial_{x_j} w - [\varphi_\nu(D_x), c(t, x)] 2^{-\alpha\nu t} w. \end{aligned}$$

Setting  $A(t, x) := (a_{i,j}(t, x))_{i,j=1}^n \in \mathcal{M}^{n \times n}$  and  $B(t, x) := (b_j(t, x))_{j=1}^n \in \mathbb{R}^n$ , we compute the scalar product of (3.3) with  $(t + \tau) \partial_t v_\nu$  and we obtain

$$\begin{aligned} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 = & \gamma(t + \tau) \langle v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ & + (t + \tau) \langle A(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x \partial_t v_\nu(t, \cdot) \rangle_{(L^2)^n} \\ & - (t + \tau) \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) \langle v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ & - \alpha(\log 2) \nu(t + \tau) \langle v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ & - (t + \tau) \langle B(t, \cdot) \cdot \nabla_x v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ & - (t + \tau) \langle c(t, \cdot) v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ & + (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2}. \end{aligned} \quad (3.4)$$

Let now  $\rho \in C_0^\infty(\mathbb{R})$ , with  $\text{supp } \rho \subseteq [-\frac{1}{2}, \frac{1}{2}]$ ,  $\int_{\mathbb{R}} \rho(s) ds = 1$  and  $\rho(s) \geq 0$  for all  $s \in \mathbb{R}$ . We set, for  $\varepsilon \in ]0, 1]$ ,

$$a_{i,j,\varepsilon}(t, x) := \int_{\mathbb{R}} a_{i,j}(s, x) \frac{1}{\varepsilon} \rho\left(\frac{t-s}{\varepsilon}\right) ds.$$

We deduce that, for all  $\varepsilon \in ]0, 1]$ ,

$$k|\xi|^2 \leq \sum_{i,j} a_{i,j,\varepsilon}(t, x) \xi_i \xi_j \leq k^{-1} |\xi|^2, \quad (3.5)$$

$$|a_{i,j,\varepsilon}(t, x) - a_{i,j}(t, x)| \leq A_{LL} \mu(\varepsilon), \quad (3.6)$$

and

$$|\partial_t a_{i,j,\varepsilon}(t, x)| \leq A_{LL} \|\rho'\|_{L^1} \frac{\mu(\varepsilon)}{\varepsilon}. \quad (3.7)$$

We set

$$a_{i,j,\nu} := a_{i,j,\varepsilon} \quad \text{with} \quad \varepsilon = 2^{-2\nu},$$

and

$$A_\nu := (a_{i,j,\nu}(t, x))_{i,j=1}^n.$$

In the second and fourth term of the right hand side part of (3.4) we replace  $A$  with  $(A - A_\nu) + A_\nu$  and  $\partial_t v_\nu$  with the quantity given by (3.3), respectively. We obtain



$$\begin{aligned}
& (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
&= \frac{d}{dt} \left( \frac{\gamma}{2} (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
&\quad + \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \langle A_\nu(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \right) \\
&\quad - \frac{1}{2} \langle A_\nu(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \\
&\quad - \frac{1}{2} (t + \tau) \langle \partial_t A_\nu(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \\
&\quad + (t + \tau) \langle (A(t, \cdot) - A_\nu(t, \cdot)) \nabla_x v_\nu(t, \cdot), \nabla_x \partial_t v_\nu(t, \cdot) \rangle_{(L^2)^n} \\
&\quad - \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) + \frac{1}{2} \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
&\quad + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 - \alpha \gamma (\log 2) \nu (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
&\quad - \alpha (\log 2) \nu (t + \tau) \langle A(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \\
&\quad + \alpha (\log 2) \nu (t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
&\quad + \alpha^2 (\log 2)^2 \nu^2 (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
&\quad + \alpha (\log 2) \nu (t + \tau) \langle v_\nu(t, \cdot), B(t, \cdot) \cdot \nabla_x v_\nu(t, \cdot) \rangle_{L^2} \\
&\quad + \alpha (\log 2) \nu (t + \tau) \langle v_\nu(t, \cdot), c(t, \cdot) v_\nu(t, \cdot) \rangle_{L^2} \\
&\quad - \alpha (\log 2) \nu (t + \tau) \langle v_\nu(t, \cdot), X_\nu(t, \cdot) \rangle_{L^2} \\
&\quad - (t + \tau) \langle B(t, \cdot) \cdot \nabla_x v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
&\quad - (t + \tau) \langle c(t, \cdot) v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\
&\quad + (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2}.
\end{aligned} \tag{3.8}$$

### 3.3 Estimate for $\nu = 0$

We consider (3.8) in the case of  $\nu = 0$ . Using Hölder inequality, the inequalities (3.6), (3.7) and the fact that  $\|\nabla_x v_0\|_{(L^2)^n} \leq 2\|v_0\|_{L^2}$  and the similar inequality

for  $\partial_t v_0$ , we deduce that, for  $t \in [0, \sigma]$ ,

$$\begin{aligned}
& (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 \\
& \leq \frac{d}{dt} \left( \frac{\gamma}{2} (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \langle A_0(t, \cdot) \nabla_x v_0(t, \cdot), \nabla_x v_0(t, \cdot) \rangle_{(L^2)^n} \right) \\
& \quad - \frac{1}{2} \langle A_0(t, \cdot) \nabla_x v_0(t, \cdot), \nabla_x v_0(t, \cdot) \rangle_{(L^2)^n} \\
& \quad + 2n A_{LL} \|\rho'\|_{L^1} (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + 32n^2 A_{LL}^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 \\
& \quad - \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \right) \\
& \quad + \frac{1}{2} \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + 8nB^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + 2C^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Choosing  $\gamma$  such that  $\frac{\gamma}{4} \geq (2nA_{LL}\|\rho'\|_{L^1} + 32n^2A_{LL}^2 + 8nB^2 + 2C^2)(\sigma + \tau)$  the term

$$\begin{aligned}
& + 2nA_{LL}\|\rho'\|_{L^1} (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + 32n^2A_{LL}^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 \\
& + 8nB^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 + 2C^2 (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2
\end{aligned}$$

is absorbed by  $-\frac{\gamma}{4} \|v_0(t, \cdot)\|_{L^2}^2$ .

Recalling that  $\Phi_\lambda$  satisfies (2.2), i. e.

$$y \Phi''_\lambda(y) = -\lambda \Phi'_\lambda(y) (1 + |\log \frac{1}{\Phi'_\lambda(y)}|),$$

with  $\lambda > 1$ , the term  $\frac{1}{2} \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2$  is balanced by  $\frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2$ .

We obtain

$$\begin{aligned}
\frac{5}{8} (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 & \leq \frac{d}{dt} \left( \frac{\gamma}{2} (t + \tau) \|v_0(t, \cdot)\|_{L^2}^2 \right) - \frac{\gamma}{4} \|v_0(t, \cdot)\|_{L^2}^2 \\
& \quad + \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \langle A_0(t, \cdot) \nabla_x v_0(t, \cdot), \nabla_x v_0(t, \cdot) \rangle_{(L^2)^n} \right) \\
& \quad - \frac{1}{2} \langle A_0(t, \cdot) \nabla_x v_0(t, \cdot), \nabla_x v_0(t, \cdot) \rangle_{(L^2)^n} \\
& \quad - \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_0(t, \cdot)\|_{L^2}^2 \right) \\
& \quad + (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Integrating the previous inequality between 0 and  $s$ , with  $s \leq \sigma$ , we have

$$\begin{aligned}
& \frac{1}{2} \int_0^s \langle A_0(t, \cdot) \nabla_x v_0(t, \cdot), \nabla_x v_0(t, \cdot) \rangle_{(L^2)^n} dt + \frac{\gamma}{8} \int_0^s \|v_0(t, \cdot)\|_{L^2}^2 dt \\
& \leq \frac{1}{2} (s + \tau) \langle A_0(s, \cdot) \nabla_x v_0(s, \cdot), \nabla_x v_0(s, \cdot) \rangle_{(L^2)^n} \\
& \quad + \frac{\gamma}{2} (s + \tau) \|v_0(s, \cdot)\|_{L^2}^2 + \frac{1}{2} \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) \|v_0(0, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\gamma}{8} \int_0^s \|v_0(t, \cdot)\|_{L^2}^2 dt - \frac{5}{8} \int_0^s (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \int_0^s (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2} dt,
\end{aligned}$$

where on the right hand side part some negative terms have been neglected. Again from the fact that  $\|\nabla_x v_0\|_{(L^2)^n} \leq 2\|v_0\|_{L^2}$ , using also (3.5), we finally deduce

$$\begin{aligned}
\frac{\gamma}{8} \int_0^s \|v_0(t, \cdot)\|_{L^2}^2 dt & \leq (2k^{-1} + \frac{\gamma}{2})(s + \tau) \|v_0(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) \|v_0(0, \cdot)\|_{L^2}^2 - \frac{\gamma}{8} \int_0^s \|v_0(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \frac{5}{8} \int_0^s (t + \tau) \|\partial_t v_0(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \int_0^s (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2} dt.
\end{aligned} \tag{3.9}$$

### 3.4 Estimate for $\nu \geq 1$

We consider (3.8) in the case of  $\nu \geq 1$ . Using again Hölder inequality, the inequalities (3.6), (3.7) and (3.1), we have that

$$\begin{aligned}
& (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \leq \frac{d}{dt} \left( \frac{\gamma}{2} (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) - \frac{\gamma}{2} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \langle A_\nu(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \right) \\
& \quad - \frac{1}{2} \langle A_\nu(t, \cdot) \nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot) \rangle_{(L^2)^n} \\
& \quad + 2(1 + 2 \log 2) n A_{LL} \|\rho'\|_{L^1} \nu(t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + 32(1 + 2 \log 2)^2 n^2 A_{LL}^2 \nu^2 (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad - \frac{d}{dt} \left( \frac{1}{2} (t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \right) + \frac{1}{2} \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \frac{t + \tau}{\beta} \Phi''_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 - \alpha \gamma (\log 2) \nu(t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad - \alpha (\log 2) \frac{k}{4} \nu(t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 + \alpha (\log 2) \nu(t + \tau) \Phi'_\lambda \left( \frac{t + \tau}{\beta} \right) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \alpha^2 (\log 2)^2 \nu^2 (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \alpha 2 (\log 2) n^{1/2} B \nu(t + \tau) 2^\nu \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + \alpha (\log 2) C \nu(t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + 32 n B^2 (t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad + 2 C^2 (t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{8} (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 \\
& \quad - \alpha (\log 2) \nu(t + \tau) \langle v_\nu(t, \cdot), X_\nu(t, \cdot) \rangle_{L^2} \\
& \quad + (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2}.
\end{aligned}$$

Let now  $\alpha = \max\{T^{-1}, \alpha_1\}$ , where

$$\alpha_1 := \frac{16}{k \log 2} (2(1 + 2 \log 2) n A_{LL} \|\rho'\|_{L^1} + 32(1 + 2 \log 2)^2 n^2 A_{LL}^2 + 32 n B^2),$$

then

$$\begin{aligned}
& -\frac{\alpha}{4} (\log 2) \frac{k}{4} \nu 2^{2\nu} + 2(1 + 2 \log 2) n A_{LL} \|\rho'\|_{L^1} \nu 2^{2\nu} \\
& \quad + 32(1 + 2 \log 2)^2 n^2 A_{LL}^2 \nu^2 + 32 n B^2 2^{2\nu} \leq 0,
\end{aligned}$$

and the term

$$\begin{aligned} & 2(1 + 2 \log 2) n A_{LL} \|\rho'\|_{L^1} \nu(t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \\ & + 32(1 + 2 \log 2)^2 n^2 A_{LL}^2 \nu^2(t + \tau) \|v_\nu(t, \cdot)\|_{L^2}^2 \\ & + 32nB^2(t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 \end{aligned}$$

is absorbed by  $-\frac{\alpha}{4}(\log 2) \frac{k}{4} \nu(t + \tau) 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2$ .

Since  $y \Phi'_\lambda(y) = -\lambda \Phi'_\lambda(y)(1 + |\log \frac{1}{\Phi'_\lambda(y)}|)$ , supposing  $\lambda > 2$  we have

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda\left(\frac{t + \tau}{\beta}\right) \leq -\frac{1}{2} \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right),$$

and the term  $\frac{1}{2} \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) \|v_\nu(t, \cdot)\|_{L^2}^2$  is absorbed by  $\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda\left(\frac{t + \tau}{\beta}\right) \|v_\nu(t, \cdot)\|_{L^2}^2$ .

Consider now the term

$$\alpha(\log 2) \nu(t + \tau) \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) \|v_\nu(t, \cdot)\|_{L^2}^2.$$

Let  $k' = \min\{k, 16\}$ . If  $\nu \geq (\log 2)^{-1} \log(\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta}))$ , then

$$-\frac{\alpha}{4}(\log 2) \frac{k}{4} \nu 2^{2\nu} \leq -\alpha(\log 2) \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) \nu.$$

On the contrary, if  $\nu < (\log 2)^{-1} \log(\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta}))$  then  $\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta}) > 2^\nu$ , so that

$$\begin{aligned} \frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda\left(\frac{t + \tau}{\beta}\right) &= -\frac{1}{4} \lambda (\Phi'_\lambda(\frac{t + \tau}{\beta}))^2 \mu\left(\frac{1}{\Phi'_\lambda(\frac{t + \tau}{\beta})}\right) \\ &\leq -\frac{1}{4} \lambda (\Phi'_\lambda(\frac{t + \tau}{\beta}))^2 \mu\left(\frac{1}{\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta})}\right) \\ &\leq -\frac{1}{4} \lambda \frac{k'}{16} \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) (1 + |\log(\frac{1}{\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta})})|) \\ &\leq -\frac{1}{4} \lambda \frac{k'}{16} \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) (1 + \nu \log 2) \\ &\leq -\lambda \frac{k' \log 2}{48} \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) \nu, \end{aligned}$$

where we have used the fact that the function  $\mu$  is increasing. Consequently if we choose  $\lambda$  in such a way that

$$\frac{\lambda k' \log 2}{48} \geq \alpha(\log 2)(\sigma + \tau), \quad \text{i. e.} \quad \lambda \geq \frac{48\alpha(\sigma + \tau)}{k'},$$

then, if  $\nu < (\log 2)^{-1} \log(\frac{16}{k'} \Phi'_\lambda(\frac{t + \tau}{\beta}))$ , we have

$$\frac{1}{4} \frac{t + \tau}{\beta} \Phi''_\lambda\left(\frac{t + \tau}{\beta}\right) \leq -\alpha(\log 2)(t + \tau) \Phi'_\lambda\left(\frac{t + \tau}{\beta}\right) \nu.$$

In conclusion  $\alpha(\log 2)\nu(t+\tau)\Phi'_\lambda(\frac{t+\tau}{\beta})\|v_\nu(t, \cdot)\|_{L^2}^2$  is balanced by  $-\frac{\alpha}{4}(\log 2)\frac{k}{4}\nu(t+\tau)2^{2\nu}\|v_\nu(t, \cdot)\|_{L^2}^2 + \frac{1}{4}\frac{t+\tau}{\beta}\Phi''_\lambda(\frac{t+\tau}{\beta})\|v_\nu(t, \cdot)\|_{L^2}^2$ . We remark that the computations here above are the main cause for the introduction of the weight function  $\Phi_\lambda$ .

Consider now the sum

$$\alpha^2(\log 2)^2(t+\tau)\nu^2\|v_\nu(t, \cdot)\|_{L^2}^2 + \alpha 2(\log 2)n^{1/2}B(t+\tau)\nu 2^\nu\|v_\nu(t, \cdot)\|_{L^2}^2.$$

If  $\nu \geq (\log 2)^{-1} \log(\frac{1}{k}(16\alpha \log 2 + 32n^{1/2}B)) =: \bar{\nu}_1$  then

$$-\frac{\alpha}{4}(\log 2)\frac{k}{4}\nu 2^{2\nu} + (\alpha^2(\log 2)^2\nu^2 + \alpha 2(\log 2)n^{1/2}B\nu 2^\nu) \leq 0.$$

If  $\nu < (\log 2)^{-1} \log(\frac{1}{k}(16\alpha \log 2 + 32n^{1/2}B)) = \bar{\nu}_1$ , choosing  $\gamma$  in such a way that

$$\frac{\gamma}{4} \geq (\alpha^2(\log 2)^2\bar{\nu}_1^2 + \alpha 2(\log 2)n^{1/2}B\bar{\nu}_1 2^{\bar{\nu}_1})(\sigma + \tau),$$

we obtain

$$-\frac{\gamma}{4} + (\alpha^2(\log 2)^2\nu^2 + \alpha 2(\log 2)n^{1/2}B\nu 2^\nu)(t+\tau) \leq 0,$$

and consequently the term  $\alpha^2(\log 2)^2(t+\tau)\nu^2\|v_\nu(t, \cdot)\|_{L^2}^2 + \alpha 2(\log 2)n^{1/2}B(t+\tau)\nu 2^\nu\|v_\nu(t, \cdot)\|_{L^2}^2$  is absorbed by  $-\frac{\alpha}{4}(\log 2)\frac{k}{4}\nu(t+\tau)2^{2\nu}\|v_\nu(t, \cdot)\|_{L^2}^2 - \frac{\gamma}{4}\|v_\nu(t, \cdot)\|_{L^2}^2$ .

Consider finally

$$\alpha(\log 2)C(t+\tau)\nu\|v_\nu(t, \cdot)\|_{L^2}^2 + 2C^2(t+\tau)\|v_\nu(t, \cdot)\|_{L^2}^2.$$

If we take  $\gamma$  such that

$$\gamma \geq \frac{\alpha(\log 2)C + 2C^2}{\alpha \log 2},$$

then

$$-\alpha\gamma(\log 2)\nu + \alpha(\log 2)C\nu + 2C^2 \leq 0,$$

and the above quoted term is absorbed by  $-\alpha\gamma(\log 2)\nu\|v_\nu(t, \cdot)\|_{L^2}^2$ .

Summing up, we set

$$\alpha_1 := \frac{16}{k \log 2}(2(1 + 2 \log 2)nA_{LL}\|\rho'\|_{L^1} + 32(1 + 2 \log 2)^2n^2A_{LL}^2 + 32nB^2),$$

$$\alpha := \max\{T^{-1}, \alpha_1\}, \quad \sigma := 1/\alpha, \quad k' := \min\{k, 16\}.$$

We choose  $\tau \in ]0, \sigma/2[$  and we define

$$\bar{\nu}_1 := (\log 2)^{-1} \log(\frac{1}{k}(16\alpha \log 2 + 32n^{1/2}B)).$$

We choose  $\lambda, \gamma$  in such a way that

$$\lambda \geq \max\{2, \frac{48\alpha(\sigma + \tau)}{k'}\},$$

$$\gamma \geq \max\left\{\frac{\alpha(\log 2)C + 2C^2}{\alpha \log 2}, 4(\alpha^2(\log 2)^2 \bar{\nu}_1^2 + \alpha 2(\log 2)n^{1/2} B \bar{\nu}_1 2^{\bar{\nu}_1})(\sigma + \tau)\right\}.$$

Then, for all  $\beta \geq \sigma + \tau$  and for all  $\nu \geq 1$  we have

$$\begin{aligned} \frac{5}{8}(t + \tau)\|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 &\leq \frac{d}{dt}\left(\frac{\gamma}{2}(t + \tau)\|v_\nu(t, \cdot)\|_{L^2}^2 - \frac{\gamma}{4}\|v_\nu(t, \cdot)\|_{L^2}^2\right. \\ &\quad + \frac{d}{dt}\left(\frac{1}{2}(t + \tau)\langle A_\nu(t, \cdot)\nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot)\rangle_{(L^2)^n}\right. \\ &\quad \left. - \frac{1}{2}\langle A_\nu(t, \cdot)\nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot)\rangle_{(L^2)^n}\right. \\ &\quad \left. - \frac{d}{dt}\left(\frac{1}{2}(t + \tau)\Phi'_\lambda\left(\frac{t + \tau}{\beta}\right)\|v_\nu(t, \cdot)\|_{L^2}^2\right)\right. \\ &\quad \left. - \frac{\alpha}{4}(\log 2)\frac{k}{4}(t + \tau)\nu 2^\nu\|v_\nu(t, \cdot)\|_{L^2}^2\right. \\ &\quad \left. - \alpha(\log 2)(t + \tau)\nu\langle v_\nu(t, \cdot), X_\nu(t, \cdot)\rangle\right. \\ &\quad \left. + (t + \tau)\langle X_\nu(t, \cdot), \partial_t v_\nu(t, \cdot)\rangle_{L^2}\right). \end{aligned}$$

We integrate this last inequality between 0 and  $s$ , with  $s \leq \sigma$ , and we obtain

$$\begin{aligned} \frac{1}{2}\int_0^s \langle A_\nu(t, \cdot)\nabla_x v_\nu(t, \cdot), \nabla_x v_\nu(t, \cdot)\rangle_{(L^2)^n} dt &+ \frac{\gamma}{8}\int_0^s \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\ &\leq \frac{1}{2}(s + \tau)\langle A_\nu(s, \cdot)\nabla_x v_\nu(s, \cdot), \nabla_x v_\nu(s, \cdot)\rangle_{(L^2)^n} \\ &\quad + \frac{\gamma}{2}(s + \tau)\|v_\nu(s, \cdot)\|_{L^2}^2 + \frac{1}{2}\tau\Phi'_\lambda\left(\frac{\tau}{\beta}\right)\|v_\nu(0, \cdot)\|_{L^2}^2 \\ &\quad - \frac{\gamma}{8}\int_0^s \|v_\nu(t, \cdot)\|_{L^2}^2 dt - \frac{5}{8}\int_0^s (t + \tau)\|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\ &\quad - \frac{\alpha k(\log 2)}{16}\int_0^s (t + \tau)\nu 2^{2\nu}\|v_\nu(t, \cdot)\|_{L^2}^2 dt \\ &\quad - \alpha(\log 2)\int_0^s (t + \tau)\nu\langle v_\nu(t, \cdot), X_\nu(t, \cdot)\rangle_{L^2} dt, \\ &\quad + \int_0^s (t + \tau)\langle X_\nu(t, \cdot), \partial_t v_\nu(t, \cdot)\rangle_{L^2} dt, \end{aligned}$$

where again on the right hand side part some negative terms have been ne-

glected. Using (3.5) and (3.1) we obtain

$$\begin{aligned}
& \frac{k}{8} \int_0^s 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq (2^{2\nu+1} k^{-1} + \frac{\gamma}{2})(s + \tau) \|v_\nu(s, \cdot)\|_{L^2}^2 + \frac{1}{2} \tau \Phi'_\lambda(\frac{\tau}{\beta}) \|v_\nu(0, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\gamma}{8} \int_0^s \|v_\nu(t, \cdot)\|_{L^2}^2 dt - \frac{5}{8} \int_0^s (t + \tau) \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \frac{\alpha k (\log 2)}{16} \int_0^s (t + \tau) \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \alpha (\log 2) \int_0^s (t + \tau) \nu \langle v_\nu(t, \cdot), X_\nu \rangle_{L^2} dt, \\
& \quad + \int_0^s (t + \tau) \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2} dt.
\end{aligned} \tag{3.10}$$

### 3.5 Estimate for the commutator term

We collect together (3.9) and (3.10). We deduce that

$$\begin{aligned}
& \frac{k}{8} \int_0^s \sum_{\nu=1}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{+\infty} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \frac{2}{k} (s + \tau) \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_\nu(s, \cdot)\|_{L^2}^2 + \frac{\gamma}{2} (s + \tau) \sum_{\nu=0}^{+\infty} \|v_\nu(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \tau \Phi'_\lambda(\frac{\tau}{\beta}) \sum_{\nu=0}^{+\infty} \|v_\nu(0, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{+\infty} \|v_\nu(t, \cdot)\|_{L^2}^2 dt - \frac{5}{8} \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \frac{\alpha k (\log 2)}{16} \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad - \alpha (\log 2) \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \nu \langle v_\nu(t, \cdot), X_\nu \rangle_{L^2} dt, \\
& \quad + \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \langle X_\nu(t, \cdot), \partial_t v_0(t, \cdot) \rangle_{L^2} dt.
\end{aligned}$$

Now we want to estimate the last two terms: to do this we shall follow essentially the ideas contained in [3]. We recall that

$$X_\nu(t, x) = X_\nu^1(t, x) + X_\nu^2(t, x) + X_\nu^3(t, x),$$



where

$$\begin{aligned} X_\nu^1(t, x) &:= - \sum_{i,j} \partial_{x_i} ([\varphi_\nu(D_x), a_{i,j}(t, x)] 2^{-\alpha\nu t} \partial_{x_j} w), \\ X_\nu^2(t, x) &:= - \sum_j [\varphi_\nu(D_x), b_j(t, x)] 2^{-\alpha\nu t} \partial_{x_j} w, \\ X_\nu^3(t, x) &:= - [\varphi_\nu(D_x), c(t, x)] 2^{-\alpha\nu t} w. \end{aligned}$$

We start with

$$\begin{aligned} & \sum_{\nu=0}^{+\infty} \langle X_\nu^1(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \\ &= \sum_{i,j} \sum_{\nu} \langle [\varphi_\nu(D_x), a_{i,j}] 2^{-\alpha\nu t} \partial_{x_j} w, \partial_{x_i} \partial_t v_\nu \rangle_{L^2} \\ &= \sum_{i,j} \sum_{\nu, \mu} \langle [\varphi_\nu(D_x), a_{i,j}] 2^{-\alpha\nu t} \partial_{x_j} w_\mu, \partial_{x_i} \partial_t v_\nu \rangle_{L^2} \\ &= \sum_{i,j} \sum_{\nu, \mu} \langle ([\varphi_\nu(D_x), a_{i,j}] \psi_\mu(D_x)) (2^{-\alpha(\nu-\mu)t} \partial_{x_j} v_\mu), \partial_{x_i} \partial_t v_\nu \rangle_{L^2} \end{aligned}$$

where  $\psi_\mu(D_x) := \varphi_{\mu-1}(D_x) + \varphi_\mu(D_x) + \varphi_{\mu+1}(D_x)$ . Consequently

$$\begin{aligned} & \left| \sum_{\nu=0}^{+\infty} \langle X_\nu^1(t, \cdot), \partial_t v_\nu(t, \cdot) \rangle_{L^2} \right| \\ & \leq \sum_{i,j} \sum_{\nu, \mu} \| [\varphi_\nu(D_x), a_{i,j}] \psi_\mu(D_x) \|_{\mathcal{L}(L^2, L^2)} 2^{-\alpha(\nu-\mu)t} 2^{\mu+1} 2^{\nu+1} \| v_\mu \|_{L^2} \| \partial_t v_\nu \|_{L^2}. \end{aligned}$$

We know that there exists  $Q_A > 0$  such that

$$\| [\varphi_\nu(D_x), a_{i,j}] \psi_\mu(D_x) \|_{\mathcal{L}(L^2, L^2)} \leq \begin{cases} 3Q_A 2^{-2\nu} & \text{if } |\mu - \nu| \leq 2, \\ 3Q_A 2^{-2 \max\{\nu, \mu\}} & \text{if } |\mu - \nu| \geq 3. \end{cases}$$

Setting  $k_{\nu, \mu}(t) := 2^{-\alpha(\nu-\mu)t} 2^\nu \| [\varphi_\nu(D_x), a_{i,j}] \psi_\mu(D_x) \|_{\mathcal{L}(L^2, L^2)}$ , for  $0 \leq t \leq 1/\alpha$ , we get, for a fixed  $\nu \geq 0$ ,

$$\begin{aligned} \sum_{\mu} |k_{\nu, \mu}(t)| & \leq \sum_{|\mu-\nu| \geq 3} 2^{-\alpha(\nu-\mu)t} 2^\nu 3Q_A 2^{-2 \max\{\nu, \mu\}} \\ & \quad + \sum_{\mu=\nu-2}^{\mu=\nu+2} 2^{-\alpha(\nu-\mu)t} 2^\nu 3Q_A 2^{-2\nu} \\ & \leq \sum_{\mu=0}^{\nu-3} 2^{-\alpha(\nu-\mu)t} 2^\nu 3Q_A 2^{-2\nu} + \sum_{\mu=\nu+3}^{+\infty} 2^{-\alpha(\nu-\mu)t} 2^\nu 3Q_A 2^{-2\mu} \\ & \quad + 3Q_A (2^{-2\alpha t} + 2^{-\alpha t} + 1 + 2^{\alpha t} + 2^{2\alpha t}) \\ & \leq 30 Q_A. \end{aligned}$$

On the other hand, for a fixed  $\nu \geq 0$ ,

$$\begin{aligned}
\sum_{\nu} |k_{\nu,\mu}(t)| &\leq \sum_{|\mu-\nu| \geq 3} 2^{-\alpha(\nu-\mu)t} 2^{\nu} 3Q_A 2^{-2 \max\{\nu,\mu\}} \\
&\quad + \sum_{\nu=\mu-2}^{\nu=\mu+2} 2^{-\alpha(\nu-\mu)t} 2^{\nu} 3Q_A 2^{-2\nu} \\
&\leq \sum_{\nu=0}^{\mu-3} 2^{-\alpha(\nu-\mu)t} 2^{\nu} 3Q_A 2^{-2\mu} + \sum_{\nu=\mu+3}^{+\infty} 2^{-\alpha(\nu-\mu)t} 2^{\nu} 3Q_A 2^{-2\nu} \\
&\quad + 3Q_A (2^{2\alpha t} + 2^{\alpha t} + 1 + 2^{-\alpha t} + 2^{-2\alpha t}) \\
&\leq 35 Q_A.
\end{aligned}$$

From Schur's criterion it follows that

$$\left| \sum_{\nu=0}^{+\infty} \langle X_{\nu}^1(t, \cdot), \partial_t v_{\nu}(t, \cdot) \rangle_{L^2} \right| \leq n^2 140 Q_A \left( \sum_{\nu=0}^{+\infty} 2^{2\mu} \|v_{\mu}\|_{L^2}^2 \right)^{1/2} \left( \sum_{\nu=0}^{+\infty} \|\partial_t v_{\mu}\|_{L^2}^2 \right)^{1/2},$$

and then, for all  $\eta > 0$ ,

$$\left| \sum_{\nu=0}^{+\infty} \langle X_{\nu}^1(t, \cdot), \partial_t v_{\nu}(t, \cdot) \rangle_{L^2} \right| \leq \frac{(n^2 140 Q_A)^2}{2\eta} \sum_{\nu=0}^{+\infty} 2^{2\mu} \|v_{\mu}\|_{L^2}^2 + \frac{\eta}{2} \sum_{\nu=0}^{+\infty} \|\partial_t v_{\mu}\|_{L^2}^2.$$

Arguing in a similar way we deduce that for all  $\eta > 0$  there exists  $Q_{\eta} > 0$  such that

$$\left| \sum_{\nu=0}^{+\infty} \langle X_{\nu}(t, \cdot), \partial_t v_{\nu}(t, \cdot) \rangle_{L^2} \right| \leq Q_{\eta} \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_{\nu}(t, \cdot)\|_{L^2}^2 + \eta \sum_{\nu=0}^{+\infty} \|\partial_t v_{\nu}(t, \cdot)\|_{L^2}^2,$$

and there exists  $\tilde{Q}_1 > 0$  such that

$$\left| \sum_{\nu=0}^{+\infty} \nu \langle v_{\nu}(t, \cdot), X_{\nu}(t, \cdot) \rangle_{L^2} \right| \leq \tilde{Q}_1 \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_{\nu}(t, \cdot)\|_{L^2}^2.$$

### 3.6 End of the proof

We have now

$$\begin{aligned}
& \frac{k}{8} \int_0^s \sum_{\nu=1}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{+\infty} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \frac{2}{k}(s + \tau) \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_\nu(s, \cdot)\|_{L^2}^2 + \frac{\gamma}{2}(s + \tau) \sum_{\nu=0}^{+\infty} \|v_\nu(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) \sum_{\nu=0}^{+\infty} \|v_\nu(0, \cdot)\|_{L^2}^2 \\
& \quad - \frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{+\infty} \|v_\nu(t, \cdot)\|_{L^2}^2 dt - \frac{5}{8} \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \quad (3.11) \\
& \quad - \frac{\alpha k (\log 2)}{16} \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \alpha (\log 2) \tilde{Q}_1 \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \int_0^s (t + \tau) (Q_\eta \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 + \eta \sum_{\nu=0}^{+\infty} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2) dt.
\end{aligned}$$

We choose  $\eta$  in such a way that  $\eta < \frac{5}{8}$  and then

$$-\frac{5}{8} \int_0^s (t + \tau) \sum_{\nu=0}^{+\infty} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt + \int_0^s (t + \tau) \eta \sum_{\nu=0}^{+\infty} \|\partial_t v_\nu(t, \cdot)\|_{L^2}^2 dt \leq 0.$$

Now if  $\nu$  is such that

$$\frac{\alpha k \log 2}{16} \nu \geq Q_\eta + \alpha (\log 2) \tilde{Q}_1,$$

then

$$\frac{\alpha k \log 2}{16} \nu 2^{2\nu} \geq Q_\eta 2^{2\nu} + \alpha (\log 2) \tilde{Q}_1 2^{2\nu}.$$

Consequently, setting  $\bar{\nu}_2 := (16/(\alpha k \log 2))(Q_\eta + \alpha (\log 2) \tilde{Q}_1)$ , we have

$$\begin{aligned}
& -\frac{\alpha k (\log 2)}{16} \int_0^s (t + \tau) \sum_{\nu=\bar{\nu}_2}^{+\infty} \nu 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + \alpha (\log 2) \tilde{Q}_1 \int_0^s (t + \tau) \sum_{\nu=\bar{\nu}_2}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \quad + Q_\eta \int_0^s (t + \tau) \sum_{\nu=\bar{\nu}_2}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \leq 0.
\end{aligned}$$

Finally, eventually choosing a larger  $\gamma$  in such a way that

$$\frac{\gamma}{8} \geq (\sigma + \tau) (\tilde{Q}_1 \alpha (\log 2) + Q_\eta) 2^{2\bar{\nu}_2},$$

we obtain

$$\begin{aligned}
& -\frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{\bar{\nu}_2-1} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \alpha(\log 2) \tilde{Q}_1 \int_0^s (t+\tau) \sum_{\nu=0}^{\bar{\nu}_2-1} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& + Q_\eta \int_0^s (t+\tau) \sum_{\nu=0}^{\bar{\nu}_2-1} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \leq 0.
\end{aligned}$$

The inequality (3.11) becomes

$$\begin{aligned}
& \frac{k}{8} \int_0^s \sum_{\nu=1}^{+\infty} 2^{2\nu} \|v_\nu(t, \cdot)\|_{L^2}^2 dt + \frac{\gamma}{8} \int_0^s \sum_{\nu=0}^{+\infty} \|v_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \frac{2}{k} (s+\tau) \sum_{\nu=0}^{+\infty} 2^{2\nu} \|v_\nu(s, \cdot)\|_{L^2}^2 + \frac{\gamma}{2} (s+\tau) \sum_{\nu=0}^{+\infty} \|v_\nu(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) \sum_{\nu=0}^{+\infty} \|v_\nu(0, \cdot)\|_{L^2}^2.
\end{aligned}$$

From this, going back to the function  $u_\nu$ , we have

$$\begin{aligned}
& \frac{k}{8} \int_0^s e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \sum_{\nu=1}^{+\infty} 2^{2\nu} 2^{-2\alpha\nu t} \|u_\nu(t, \cdot)\|_{L^2}^2 dt \\
& + \frac{\gamma}{8} \int_0^s e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \sum_{\nu=0}^{+\infty} 2^{-2\alpha\nu t} \|u_\nu(t, \cdot)\|_{L^2}^2 dt \\
& \leq \frac{2}{k} (s+\tau) e^{2\gamma s} e^{-2\beta\Phi_\lambda(\frac{s+\tau}{\beta})} \sum_{\nu=0}^{+\infty} 2^{2\nu} 2^{-2\alpha\nu s} \|u_\nu(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{\gamma}{2} (s+\tau) e^{2\gamma s} e^{-2\beta\Phi_\lambda(\frac{s+\tau}{\beta})} \sum_{\nu=0}^{+\infty} 2^{-2\alpha\nu s} \|u_\nu(s, \cdot)\|_{L^2}^2 \\
& \quad + \frac{1}{2} \tau \Phi'_\lambda\left(\frac{\tau}{\beta}\right) e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \sum_{\nu=0}^{+\infty} \|u_\nu(0, \cdot)\|_{L^2}^2,
\end{aligned}$$

and (2.5) follows immediately from (3.2), concluding the proof of Theorem 1.

## 4 Proofs of Theorems 2 and 3

We start with a lemma that will be used in the proof of Theorem 2.

**Lemma 2.** *Let  $u \in \mathcal{E}$  be a solution of equation (2.1). Then there exists  $\gamma_0 > 0$  such that if  $\gamma > \gamma_0$  then the function  $E(t) := e^{2\gamma t} \|u(t, \cdot)\|_{L^2}^2$  is (weakly) increasing.*

*Proof.* It is sufficient to compute the derivative of  $E(t)$ . We obtain

$$\begin{aligned}
\frac{d}{dt}(e^{2\gamma t}\|u(t, \cdot)\|_{L^2}^2) &= 2\gamma e^{2\gamma t}\|u(t, \cdot)\|_{L^2}^2 + 2e^{2\gamma t}\langle u(t, \cdot), \partial_t u(t, \cdot) \rangle_{L^2} \\
&= 2\gamma e^{2\gamma t}\|u(t, \cdot)\|_{L^2}^2 + 2e^{2\gamma t}\langle A(t, \cdot)\nabla_x u(t, \cdot), \nabla_x u(t, \cdot) \rangle_{(L^2)^n} \\
&\quad - 2e^{2\gamma t}(\langle B(t, \cdot)\nabla_x u(t, \cdot), u(t, \cdot) \rangle_{L^2} + \langle c(t, \cdot)u(t, \cdot), u(t, \cdot) \rangle_{L^2}) \\
&\geq 2\gamma e^{2\gamma t}\|u(t, \cdot)\|_{L^2}^2 + 2e^{2\gamma t}k\|\nabla_x u(t, \cdot)\|_{(L^2)^n}^2 \\
&\quad - 2e^{2\gamma t}n^{1/2}B\|\nabla_x u(t, \cdot)\|_{(L^2)^n}\|u(t, \cdot)\|_{L^2} - 2e^{2\gamma t}C\|u(t, \cdot)\|_{L^2}^2,
\end{aligned}$$

and the conclusion follows easily.  $\square$

Let us come to the proof of Theorem 2. Let  $\sigma, \bar{\lambda}, \alpha, \bar{\gamma}, M$  as in Theorem 1. We choose  $\lambda \geq \bar{\lambda}$  and  $\gamma \geq \max\{\bar{\gamma}, \gamma_0\}$  where  $\gamma_0$  is given by Lemma 2. We set  $\tau = \frac{\sigma}{2} - 2\bar{\sigma}$  (we recall that  $\bar{\sigma} \in ]0, \sigma/4[$  and then  $\frac{\sigma}{2} - 2\bar{\sigma} \in ]0, \sigma/2[$ ). Then (2.5) gives

$$\begin{aligned}
&\int_0^\sigma e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{H^{1-\alpha t}}^2 dt \\
&\leq M((\sigma + \tau)e^{2\gamma\sigma} e^{-2\beta\Phi_\lambda(\frac{\sigma+\tau}{\beta})} \|u(\sigma, \cdot)\|_{L^2}^2 + \tau\Phi'_\lambda(\frac{\tau}{\beta})e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2),
\end{aligned}$$

for all  $\beta \geq \sigma + \tau$ . Let now  $s \in [0, \bar{\sigma}]$ . Then  $2s + \tau \leq 2\bar{\sigma} + \tau \leq \frac{\sigma}{2} < \sigma$  and consequently

$$\begin{aligned}
&\int_s^{2s+\tau} e^{2\gamma t} e^{-2\beta\Phi_\lambda(\frac{t+\tau}{\beta})} \|u(t, \cdot)\|_{L^2}^2 dt \\
&\leq M((\sigma + \tau)e^{2\gamma\sigma} e^{-2\beta\Phi_\lambda(\frac{\sigma+\tau}{\beta})} \|u(\sigma, \cdot)\|_{L^2}^2 + \tau\Phi'_\lambda(\frac{\tau}{\beta})e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2),
\end{aligned}$$

where we have used the fact that  $\|u(t, \cdot)\|_{L^2} \leq \|u(t, \cdot)\|_{H^{1-\alpha t}}$ . From Lemma 2  $\|u(t, \cdot)\|_{L^2}$  is increasing. Also the function  $\Phi_\lambda$  is increasing and consequently the function  $t \mapsto e^{-2\beta\Phi_\lambda((t+\tau)/\beta)}$  is decreasing. We deduce that

$$\begin{aligned}
&e^{2\gamma s} e^{-2\beta\Phi_\lambda(\frac{2s+\tau}{\beta})} (s + \tau) \|u(s, \cdot)\|_{L^2}^2 \\
&\leq M((\sigma + \tau)e^{2\gamma\sigma} e^{-2\beta\Phi_\lambda(\frac{\sigma+\tau}{\beta})} \|u(\sigma, \cdot)\|_{L^2}^2 + \tau\Phi'_\lambda(\frac{\tau}{\beta})e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2).
\end{aligned}$$

Then

$$\begin{aligned}
\|u(s, \cdot)\|_{L^2}^2 &\leq M\left(\frac{\sigma + \tau}{\tau} e^{2\gamma\sigma} \Phi'_\lambda\left(\frac{\tau}{\beta}\right) \left( e^{2\beta(\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - \Phi_\lambda(\frac{\sigma+\tau}{\beta}))} \|u(\sigma, \cdot)\|_{L^2}^2 \right. \right. \\
&\quad \left. \left. + e^{2\beta(\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - \Phi_\lambda(\frac{\tau}{\beta}))} \|u(0, \cdot)\|_{L^2}^2 \right) \right) \\
&\leq \tilde{M} \Phi'_\lambda\left(\frac{\tau}{\beta}\right) e^{2\beta(\Phi_\lambda(\frac{\sigma/2+\tau}{\beta}) - \Phi_\lambda(\frac{\sigma+\tau}{\beta}))} \left( \|u(\sigma, \cdot)\|_{L^2}^2 \right. \\
&\quad \left. + e^{-2\beta\Phi_\lambda(\frac{\tau}{\beta})} \|u(0, \cdot)\|_{L^2}^2 \right),
\end{aligned}$$

where  $\tilde{M}$  depends on  $\sigma, \tau, \gamma$  and  $M$ . We recall that the function  $\Phi_\lambda$  is concave, so that

$$\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right) - \Phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right) \leq \Phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\left(\frac{\sigma/2+\tau}{\beta} - \frac{\sigma+\tau}{\beta}\right) = -\Phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\frac{\sigma}{2\beta},$$

and then

$$\|u(s, \cdot)\|_{L^2}^2 \leq \tilde{M}\Phi'_\lambda\left(\frac{\tau}{\beta}\right)e^{-\sigma\Phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}(\|u(\sigma, \cdot)\|_{L^2}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0, \cdot)\|_{L^2}^2).$$

From (2.3) we have that

$$\Phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right) = \psi_\lambda\left(\frac{\sigma+\tau}{\tau}\frac{\tau}{\beta}\right) = \exp\left(\left(\frac{\sigma+\tau}{\tau}\right)^{-\lambda} - 1\right)\left(\psi_\lambda\left(\frac{\tau}{\beta}\right)\right)^{\left(\frac{\sigma+\tau}{\tau}\right)^{-\lambda}}.$$

Then, setting  $\tilde{\delta} := ((\sigma + \tau)/\tau)^{-\lambda}$  we obtain that there exists  $\tilde{N} > 0$  such that

$$\|u(s, \cdot)\|_{L^2}^2 \leq \tilde{M}\psi_\lambda\left(\frac{\tau}{\beta}\right)\exp(-\tilde{N}(\psi_\lambda\left(\frac{\tau}{\beta}\right))^{\tilde{\delta}})(\|u(\sigma, \cdot)\|_{L^2}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0, \cdot)\|_{L^2}^2).$$

We choose now  $\beta$  in such a way that  $e^{-\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} = \|u(0, \cdot)\|_{L^2}^{-1}$  i. e.

$$\frac{\beta}{\tau}\Phi_\lambda\left(\frac{\tau}{\beta}\right) = \frac{1}{\tau}\log\|u(0, \cdot)\|_{L^2}.$$

We obtain  $\beta = \tau\Lambda^{-1}(\frac{1}{\tau}\log\|u(0, \cdot)\|_{L^2})$  and then there exists  $\bar{\rho} > 0$  such that if  $\|u(0, \cdot)\|_{L^2} \leq \bar{\rho}$ , then  $\beta \geq \sigma + \tau$ .

Finally we have

$$\|u(s, \cdot)\|_{L^2}^2 \leq \tilde{M}\exp\left(-\frac{\tilde{N}}{2}\left[\psi_\lambda\left(\frac{1}{\Lambda^{-1}(\frac{1}{\tau}\log\|u(0, \cdot)\|_{L^2})}\right)\right]^{\tilde{\delta}}\right)(\|u(\sigma, \cdot)\|_{L^2}^2 + 1),$$

and, from (2.4),

$$\|u(s, \cdot)\|_{L^2}^2 \leq \tilde{M}\exp\left(-\tilde{N}\left[\frac{1}{\tau}\log\|u(0, \cdot)\|_{L^2}\right]^{\tilde{\delta}}\right)(\|u(\sigma, \cdot)\|_{L^2}^2 + 1).$$

The inequality (2.6) easy follows, concluding the proof of Theorem 2.

To prove Theorem 3 it is sufficient to iterate a finite number of times the local result of Theorem 2 choosing for instance  $\bar{\sigma} = \sigma/8$ .

## Appendix

In the construction of the following example we will follow closely [12] (see also [4]). Let  $A, B, C, J$  be four  $C^\infty$  functions defined in  $\mathbb{R}$  with  $0 \leq A(s), B(s), C(s)$

$\leq 1$ ,  $-2 \leq J(s) \leq 2$  for all  $s \in \mathbb{R}$  and

$$\begin{aligned} A(s) &= 1 \quad \text{for all } s \leq \frac{1}{5}, & A(s) &= 0 \quad \text{for all } s \geq \frac{1}{4}, \\ B(s) &= 0 \quad \text{for all } s \leq 0 \text{ or } s \geq 1, & B(s) &= 1 \quad \text{for all } \frac{1}{6} \leq s \leq \frac{1}{2}, \\ C(s) &= 0 \quad \text{for all } s \leq \frac{1}{4}, & C(s) &= 1 \quad \text{for all } s \geq \frac{1}{3}, \\ J(s) &= -2 \quad \text{for all } s \leq \frac{1}{6} \text{ or } s \geq \frac{1}{2}, & J(s) &= 2 \quad \text{for all } \frac{1}{5} \leq s \leq \frac{1}{3}. \end{aligned}$$

Let  $(a_n)_n, (z_n)_n$  be two real sequences such that

$$0 \leq a_n < a_{n+1} \quad \text{for all } n \geq 1 \quad \text{and} \quad \lim_n a_n = +\infty, \quad (\text{A.1})$$

$$1 \leq z_n < z_{n+1} \quad \text{for all } n \geq 1 \quad \text{and} \quad \lim_n z_n = +\infty. \quad (\text{A.2})$$

Let us define  $r_n = a_{n+1} - a_n$ ,  $q_1 = 0$ ,  $q_n = \sum_{k=2}^n z_k r_{k-1}$  for all  $n \geq 2$ , and  $p_n = (z_{n+1} - z_n)r_n$ . We suppose that

$$r_n < 1 \quad \text{for all } n \geq 1, \quad (\text{A.3})$$

$$p_n > 1 \quad \text{for all } n \geq 1. \quad (\text{A.4})$$

We set  $A_n(t) = A(\frac{t-a_n}{r_n})$ ,  $B_n(t) = B(\frac{t-a_n}{r_n})$ ,  $C_n(t) = C(\frac{t-a_n}{r_n})$  and  $J_n(t) = J(\frac{t-a_n}{r_n})$ . We define

$$\begin{aligned} v_n(t, x_1) &= \exp(-q_n - z_n(t - a_n)) \cos \sqrt{z_n} x_1, \\ w_n(t, x_2) &= \exp(-q_n - z_n(t - a_n) + J_n(t)p_n) \cos \sqrt{z_n} x_2, \end{aligned}$$

and, for  $n_0 \geq 1$  to be chosen,

$$u(t, x_1, x_2) = v_{n_0}(t, x_1)$$

for all  $t \leq a_{n_0}$ ,  $(x_1, x_2) \in \mathbb{R}^2$  and

$$u(t, x_1, x_2) = A_n(t)v_n(t, x_1) + B_n(t)w_n(t, x_2) + C_n(t)v_{n+1}(t, x_1)$$

for all  $n \geq n_0$ ,  $a_n \leq t \leq a_{n+1}$  and  $(x_1, x_2) \in \mathbb{R}^2$ . If, for all  $\alpha, \beta, \gamma > 0$ ,

$$\lim_n \exp(-q_n + 2p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0, \quad (\text{A.5})$$

then  $u$  is a  $C_B^\infty(\mathbb{R}^3)$  function, where  $C_B^\infty$  denotes the smooth functions which are bounded with bounded derivatives. We define

$$l(t) = \begin{cases} 1 & \text{for all } t \leq a_1, \\ 1 + J'_n(t)p_n z_n^{-1} & \text{for all } a_n \leq t \leq a_{n+1}. \end{cases}$$

The condition

$$\sup_{n \geq n_0} \{p_n r_n^{-1} z_n^{-1}\} \leq \frac{1}{2\|J'\|_{L^\infty}} \quad (\text{A.6})$$

guarantees that the operator  $L = \partial_t - \partial_{x_1}^2 - l(t)\partial_{x_2}^2$  is parabolic. The function  $l$  is smooth and it is Log-Lipschitz continuous on  $\mathbb{R}$  (i. e.

$$\sup_{t_1 < t_2} \frac{|l(t_2) - l(t_1)|}{|t_2 - t_1|(|\log|t_2 - t_1|| + 1)} < +\infty)$$

under the following condition

$$\sup_n \left\{ \frac{p_n r_n^{-1} z_n^{-1}}{r_n \log(\frac{1}{r_n})} \right\} < +\infty. \quad (\text{A.7})$$

Finally we define

$$\begin{aligned} b_1 &= -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_1} u, \\ b_2 &= -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_2} u, \\ c &= -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} u \end{aligned}$$

and, as in [12], the coefficients  $b_1, b_2, c$  will be in  $C_B^\infty(\mathbb{R}^3)$  if, for all  $\alpha, \beta, \gamma > 0$ ,

$$\lim_n \exp(-p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0. \quad (\text{A.8})$$

We set

$$a_1 = 0, \quad a_n = \sum_{j=2}^n \frac{1}{j \log j} \quad \text{for all } n \geq 2,$$

and

$$z_n = n^4 \quad \text{for all } n \geq 1.$$

With these choices the conditions (A.1), (A.2), (A.3) and (A.4) are trivial and also (A.5), (A.7) and (A.8) are easily verified. From (A.7) and the fact that  $\lim_n r_n = 0$  we deduce that

$$\lim_n p_n r_n^{-1} z_n^{-1} = 0,$$

consequently it is possible to choose  $n_0$  such that (A.6) is verified.

Let now

$$n_{1,k} = [\exp(\exp(k))] + 2 \quad \text{and} \quad n_{2,k} = [\exp(\exp(k + \frac{1}{k}))] + 1,$$

where  $[x]$  denotes the integer part of  $x$  and where  $k$  is taken in such a way that  $n_{1,k} \geq n_0$ . We fix, for  $h = 1, 2$ ,

$$t_{h,k} = a_{n_{h,k}} = \sum_{j=1}^{n_{h,k}-1} r_j,$$



We have  $\lim_k t_{1,k} = +\infty$  and

$$t_{2,k} - t_{1,k} = \sum_{j=n_{1,k}}^{n_{2,k}-1} r_j \leq \int_{n_{1,k}-1}^{n_{2,k}-1} \frac{1}{x \log x} dx \leq \int_{\exp(\exp(k))}^{\exp(\exp(k+\frac{1}{k}))} \frac{1}{x \log x} dx = \frac{1}{k},$$

so that  $\lim_k t_{2,k} - t_{1,k} = 0$ . Our intent is to prove that

$$\lim_k \frac{\exp(-q_{n_{1,k}})}{\exp(-\delta q_{n_{2,k}})} = +\infty \quad (\text{A.9})$$

for all  $\delta \in (0, 1)$ . Since, by the choice of  $(a_n)_n$  and  $(z_n)_n$  we have that  $q_n = \sum_{j=2}^n j^3 / \log j$ , it is immediate to obtain that

$$q_{n_{2,k}} \geq q_{n_{1,k}} + \frac{n_{1,k}^3}{\log(n_{1,k})} (n_{2,k} - n_{1,k}),$$

and (A.9) is a consequence of

$$\lim_k \delta (q_{n_{1,k}} + \frac{n_{1,k}^3}{\log(n_{1,k})} (n_{2,k} - n_{1,k})) - q_{n_{1,k}} = +\infty.$$

This result will be implied by

$$\lim_k \delta \frac{n_{1,k}^3}{\log(n_{1,k})} (n_{2,k} - n_{1,k}) - q_{n_{1,k}} = +\infty, \quad (\text{A.10})$$

for all  $\delta \in (0, 1)$ . Easily we have that  $q_{n_{1,k}} \leq n_{1,k}^4$  for all  $k$  and consequently (A.10) may be deduced from

$$\lim_k \delta' \frac{n_{1,k}^3 n_{2,k}}{\log(n_{1,k})} - n_{1,k}^4 = +\infty$$

i. e.

$$\lim_k \delta' \frac{n_{2,k}}{\log(n_{1,k})} - n_{1,k} = +\infty \quad \text{for all } \delta' \in (0, 1),$$

which can be elementary obtained substituting  $n_{1,k}$  and  $n_{2,k}$  with their values. Summing up we have the following result.

**Theorem 4.** *There exist*

- $l \in C^\infty(\mathbb{R})$ ,  $l$  Log-Lipschitz continuous,  $1/2 \leq l(t) \leq 3/2$  for all  $t \in \mathbb{R}$ ,
- $b_1, b_2, c$  and  $u \in C_B^\infty(\mathbb{R}^3)$ ,  $2\pi$ -periodic with respect to  $x_1$  and  $x_2$ ,
- $(t_{1,n})_n, (t_{2,n})_n$  increasing sequences in  $\mathbb{R}$ ,  $1 > t_{2,n} - t_{1,n} > 0$  for all  $n$ ,  $\lim_n t_{1,n} = +\infty$  and  $\lim_n t_{2,n} - t_{1,n} = 0$ ,

such that

$$\partial_t u - \partial_{x_1}^2 u - l \partial_{x_2}^2 u + b_1 \partial_{x_1} u + b_2 \partial_{x_2} u + cu = 0$$

for all  $(t, x_1, x_2) \in \mathbb{R}^3$  and

$$\lim_n \frac{\|u(t_{1,n}, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}}{\|u(t_{2,n}, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}^\delta} = +\infty$$

for all  $\delta \in (0, 1)$ .

We define now, for  $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2$ ,

$$l_n(t) = l(t_{2,n} - t),$$

$$u_n(t, x_1, x_2) = u(t_{2,n} - t, x_1, x_2),$$

and similarly for  $b_{1,n}$ ,  $b_{2,n}$   $c_n$ . We set  $t_n = t_{2,n} - t_{1,n}$  and

$$L_n = \partial_t + \partial_{x_1}^2 + l_n \partial_{x_2}^2 u - b_{1,n} \partial_{x_1} - b_{2,n} \partial_{x_2} - c_n.$$

We have that  $(L_n)_n$  is a sequence of uniformly backward parabolic operators with uniformly Log-Lipschitz continuous coefficients in the principal part and uniformly bounded coefficients in lower order terms.  $(u_n)_n$  is a sequence of smooth uniformly bounded solutions of  $L_n u_n = 0$  on  $[0, 1] \times \mathbb{R}^2$ , with

$$\lim_n \|u_n(0, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])} = 0.$$

We have that  $\lim_n t_n = 0$  and

$$\lim_n \frac{\|u_n(t_n, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}}{\|u_n(0, \cdot, \cdot)\|_{L^2([0, 2\pi] \times [0, 2\pi])}^\delta} = +\infty$$

for all  $\delta \in (0, 1)$ : it is not possible to obtain a result similar to that of Hurd [6] or Agmon and Nirenberg [1] if Lipschitz continuity is replaced by Log-Lipschitz continuity.

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